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On the Equivalence of Some Exact Master Equations

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In this note we show the equivalence of two procedures for obtaining exact, convolutionless master equations.

KEY WORDS: Exact; convolutionless master equations.

The purpose of this note is to reconcile two seemingly different sets of exact, convolutionless master equations. On the one hand there are the equations obtained by Fulinski and Kramarczyk in Ref. 1 and on the other the equations obtained by Hashitome, Shibata and Shingū in Ref. 3. Both groups obtain the "evolution equations for the relevant part of the density matrix" by a similar procedure, both extending previous work (quoted in their respective lists of references) and overcoming the usual problems of apparent non-Markovianness. Moreover, in another paper,[(2)] Fulinski showed the equivalence of his procedure to the traditional procedure of obtaining master equations.

The setup of both Ref. 1 and Ref. 3 consists of a one-parameter group (or semigroup) $U(t)$ of operators, that we can consider acting on an algebra of operators, a Banach space, a convex set of a Banach space, etc. Let us denote the domain of $U(t)$ by B.

Let L denote the infinitesimal generator of $U(t)$. We assume that L is time independent and then $U(t)$ satisfies

$$\frac{dU}{dt} = \dot{U}(t) = LU(t), \qquad U(0) = I \tag{1.1}$$

When B is an algebra, in some cases L is taken to be a derivation of the type

$$L(A) = [H, A]$$

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where H is given element in B and A varies in B. Also for appropriate Hermiticity conditions on L (or H) $U(t)$ will be a unitary one-parameter (semi-) group. But this is irrelevant for our analysis.

In kinetic theory one is interested in the evolution equation of a relevant part of $\rho(t) = U(t)\rho_0$, ρ_0 in B. The quantity $\rho(t)$ evolves according to

$$\dot{\rho}(t) = L\rho(t), \qquad \rho(0) = \rho_0 \tag{1.2}$$

and from this equation one is interested in obtaining an evolution equation for $P\rho(t)$, where $P^2 = P$ is some projection operator. Put $Q = 1 - P$.

Let us rapidly recall the basics in Refs. 1 and 3 in our notation. For that purpose let

$$N(t) = Q + PU(t) = 1 + P(U(t) - 1) \tag{1.3}$$

It is shown in Ref. 1 that $N(t)^{-1}$ exists and that

$$\begin{aligned} P\dot{\rho}(t) &= PL\rho(t) = PLU(t)N(t)^{-1}N(t)\rho_0 \\ &= PLU(t)N(t)^{-1}(Q\rho_0 + P\rho_t) \end{aligned} \tag{1.4}$$

which is an exact convolutionless master equation for $P\rho(t)$.

Now, instead of (1.3) let us put

$$M(t) = P + Z(t)\,QU(t)^{-1} \tag{1.5}$$

where $Z(t) = \exp(tQL)$ and $U(t)^{-1} = \exp(-tL)$. The same proof as in Ref. 1 yields the existence of an inverse to $M(t)$. Consider now

$$P\dot{\rho}(t) = PLM(t)^{-1}M(t)U(t)\rho_0 = PLM(t)^{-1}[P\rho(t) + Z(t)Q\rho_0] \tag{1.6}$$

which is the form obtained in Ref. 3.

Certainly, if there is any justice in the world, (1.6) and (1.4) should be the same. Let us verify it. Observe that

$$\begin{aligned} M(t)U(t) &= PU(t) + Z(t)\,Q = PU(t) + Q + [Z(t) - 1]\,Q \\ &= N(t) + [Z(t) - 1]\,Q \end{aligned}$$

or equivalently

$$N(t) = M(t)U(t) + [1 - Z(t)]\,Q$$

from which it follows that $N(t)^{-1}$ satisfies the equation

$$N(t)^{-1} = U(t)^{-1}M(t)^{-1} - U(t)^{-1}M(t)^{-1}[1 - Z(t)]\,QN(t)^{-1} \tag{1.7}$$

Since $QN(t)^{-1} = Q$, because

$$N(t)^{-1} = \sum_0^{\infty} \{-P(U(t) - 1)\}^n$$

and $QP = 0$, we obtain

$$U(t)N(t)^{-1} = M(t)^{-1}[P + Z(t)Q]$$

and since also $PQ = 0$ it follows that

$$\begin{aligned} LU(t)^{-1}N(t)^{-1}[Q\rho_0 + P\rho(t)] &= LM(t)^{-1}[P + Z(t)Q][Q\rho_0 + P\rho(t)] \\ &= LM(t)^{-1}[Z(t)Q\rho_0 + P\rho(t)] \end{aligned}$$

Applying P to both sides we see that (1.6) and (1.4) are the same equation in a different form. The obvious concluding remarks are: the convenience in the choice of (1.4) or (1.6) for a given problem will depend on the computability of $N(t)^{-1}$ or $M^{-1}(t)$, the integrability of (1.4) or (1.6), or on which has easier to handle perturbative expansions.

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